بسم الله الرحمن الرحيم

Lectures 1-5

Course Name: Analytical Mechanics **Phys 252**

Instructor Name: Dr. Hala A Al-Jawhari Room No. (108/C).

Description of the course: The material covered in this course is concerned with Newton's laws of motion and their applications to linear motion, harmonic oscillator, and inertial forces,

The textbook: ***Analytical Mechanics***
by G.R. Fowles & G.R. Cassiday. 7th ed.

Grading:

H.W	10%	On each tutorial session
Quiz	10%	After each tutorial session
1st EXAM	20%	week 7 **25/4/1431 H**
2nd EXAM	20%	week 12 **8/6/1431 H**
FINAL EXAM	40%	End of the semester

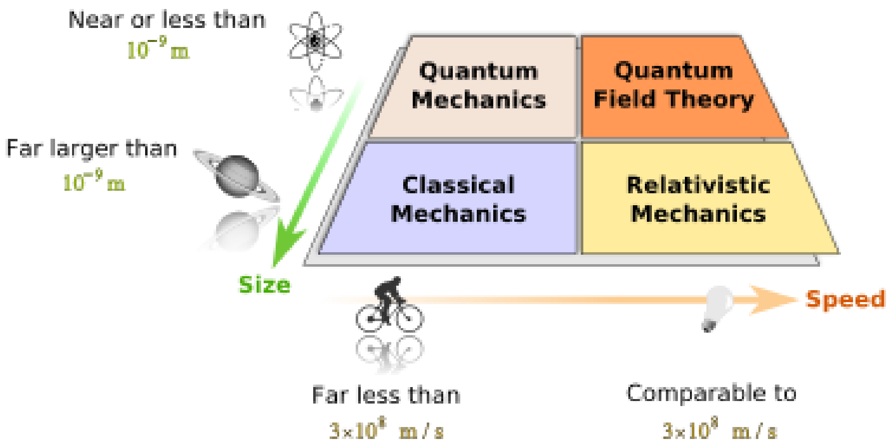
Basic Definitions

What is Mechanics?

Mechanics is ***the science of motion***, which is a change in the position of a body with respect to time. It studies the interactions between a moving body and the forces acting on it. Hence:

Classical Mechanics:

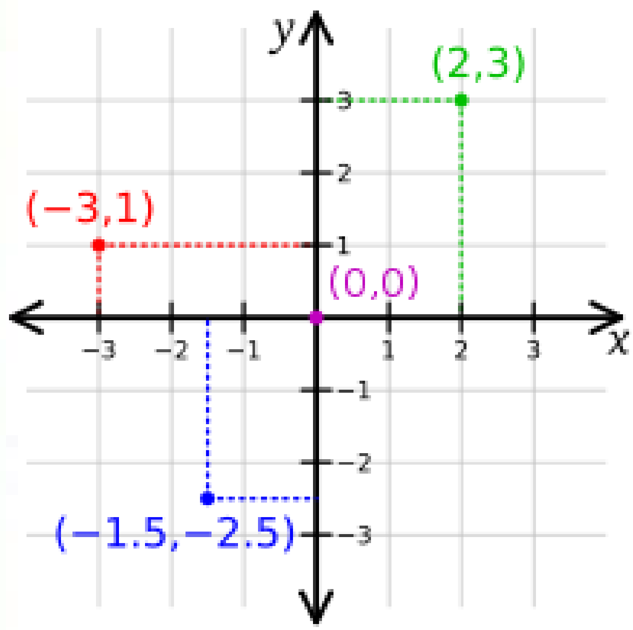
Is the science Deals with the motion of objects through absolute space & time in the Newtonian sense.

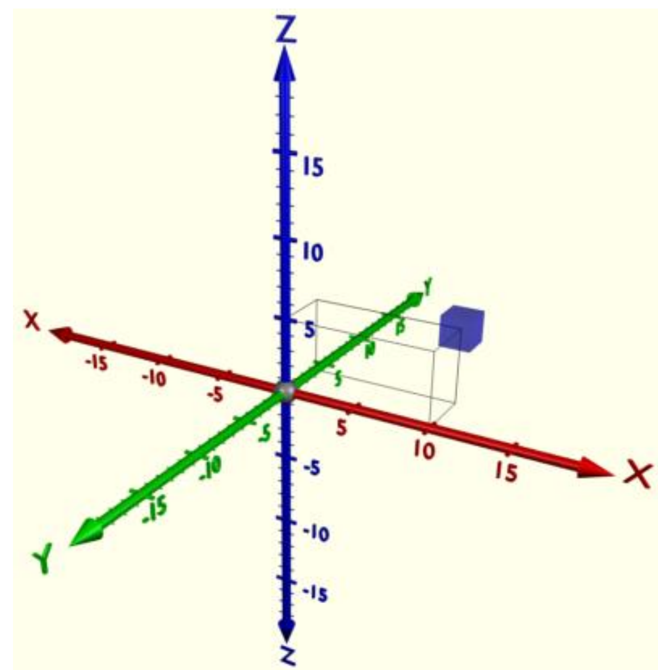


Basic Definitions

The Coordinate System:

In order to define the position of a body in space, it is necessary to have ***a reference system***. In mechanics we use a coordinate system. The basic type of coordi-nate system is the *Cartesian or rectangular coordinate system*. The position of a point in such system is specified by three numbers or coordinates, **x, y,** and **z**. In case of a moving point, the coordinates change with time; that is, **x, y,** and **z** are functions of the quantity **t**.





The particle (or mass point):

The particle is ***a thing that has mass but does not have spatial extension***. It is an approximation of a very small body, or one whose size is relatively unimportant in a particular discussion. The earth, for example, might be treated as a particle in *Celestial mechanics*.

Physical Quantities:

The observational data of physics are expressed in terms of certain fundamental entities called *physical quantities*, for example; energy, time, force, and so on.
It has been found that it is possible to define all of the physical quantities of mechanics in terms of **three basic ones**, namely ***length, time*** *and* ***mass***.

Basic Units in (SI System):

1- **meter (m)**.

2- **second (s)**.

3- **kilogram (kg)**.

prefixes of the SI Units:

Factor	10^9	10^6	10^3	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}
Prefix	Giga-	Mega-	Kilo-	Centi-	Milli-	Micro-	Nano-	Pico-
Symbol	**G**	**M**	**K**	**c**	**m**	**μ**	**n**	**p**

Dimensions:

The dimension of any physical quantity can be written as $[M]^{\alpha}\,[L]^{\beta}\,[T]^{\gamma}$ where α, β, and γ are powers of their respective dimension. For example, the dimension of acceleration **a** is

$$\boxed{a = \left[\frac{L/T}{T}\right] = [L][T]^{-2}}$$

Dimensional Analysis:

1- ***Dimensional analysis*** is a powerful tool that can be used to determine whether the result of a calculation has even the possibility of being correct or not.

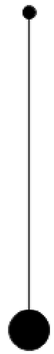
I.e.

the dimensions on the LHS = the dimension of all physical quantities on the RHS.

Dimensional Analysis:

Example:

2- ***Dimensional analysis*** can also be used as to obtain a rough relationship between physical quantities.



Consider the simple pendulum; that consists of a small mass ***m*** attached to the end of a massless string of length ***l***.

The question now is:
In the absence of friction, air resistance and all other dissipative forces, **how does the pendulum period τ depend on any physical parameters** that characterize the pendulum?

Let assume that $\tau \propto m^{\alpha} l^{\beta} g^{\gamma}$

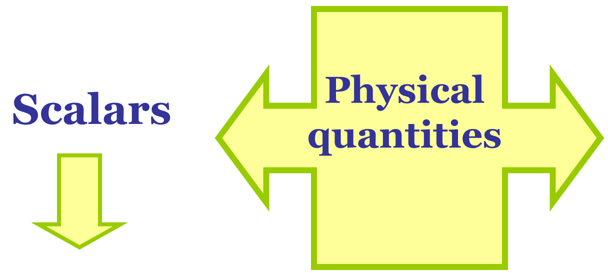
whose dimensional relationship must be

$$[T]^{1} = [M]^{\alpha} [L]^{\beta} ([L]^{\gamma} [T]^{-2\gamma})$$

Matching the dimensions of RHS with LHS we find that;
$\alpha = 0$, $\gamma = -1/2$ and, $\beta + \gamma = 0$, or $\beta = 1/2$.
Thus, we conclude that

$$\boxed{\tau \propto \sqrt{\frac{l}{g}}}$$

Vectors



- A scalar can be completely defined by *magnitude*.
- Examples: mass, density, volume, and energy.
- Its value is independent of any chosen coordinates.
- Mathematically, scalars obey the normal algebraic rules of addition, multiplication …etc.

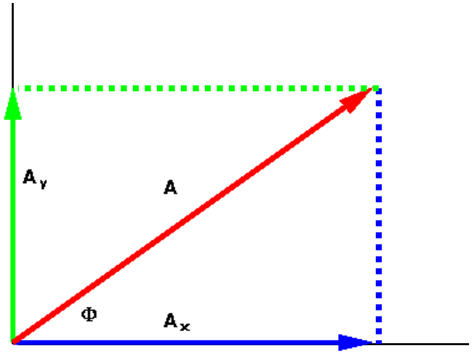
- A vector needs both *magnitude* and *direction* to be defined.
- Examples: displacement, velocity, acceleration, and force.
- Its value is coordinate system dependent.
- Mathematically, vectors need a special treatment known as *Vectors Algebra*.

Vector Algebra

How to define a Vector ?

A given vector **A** can be specified either by:

1- Its *magnitude* (A) and its *direction* (ϕ) relative to some chosen coordinate system.



2- The set of its ***components***, or projections onto the coordinate axes ; Since

and

$$A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ ***The magnitude***

$$\phi = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$ ***The direction***

The Null Vector:

The vector **O** = (0,0,0) is called the *null vector*.

Vector Addition:

The addition of two vectors is defined by the equation:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z)$$

Note that:

$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ ***Commutative Law***

$\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$ ***Associative Law***

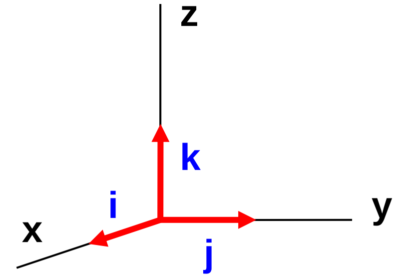
The Unit Vectors:

A unit vector is a vector whose magnitude is **unity**. Unit vectors are often assigned by the symbol **e**, The three unit vectors

$\mathbf{e}_x = (1,0,0)$ $\mathbf{e}_y = (0,1,0)$ $\mathbf{e}_z = (0,0,1)$

for ***Cartesian coordinates***

$\mathbf{e}_x = \mathbf{i}$ $\mathbf{e}_y = \mathbf{j}$ $\mathbf{e}_z = \mathbf{k}$



The Scalar Product:

Given two vectors **A** and **B**, the ***scalar or " dot " product***, **A.B** is the scalar defined by the equation

$$\boxed{\mathbf{A.B} = A_xB_x + A_yB_y + A_zB_z}$$

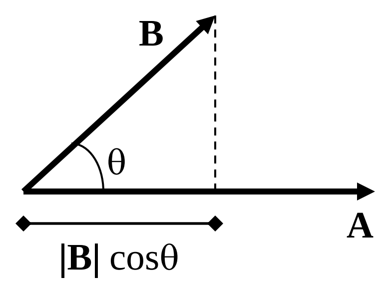
Main properties of the scalar product:

- It is ***commutative***, **A.B = B.A**
- It is ***distributive,*** **A . (B + C) = A . B + A . C**
- It is a ***scalar***. If vector **A** is expressed as $(A, 0, 0)$ and the vector **B** as $(B \cos\theta, B \sin\theta, 0)$, then,

$$\boxed{\mathbf{A.B} = A_xB_x = A(B \cos \theta) = |\mathbf{A}|\ |\mathbf{B}| \cos \theta}$$

- Hence, The geometrical interpretation of **A . B** is that it is the projection of **B** onto **A** times the length of **A**.

- If $\mathbf{A.\ B} = 0$, and neither **A** nor **B** is null, then ($\cos\theta = 0$) and **A** is ⊥ to **B**.



Similarly;

$$\boxed{\begin{array}{c}\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1 \\ \mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0\end{array}}$$

The Vector Product

Given two vectors **A** and **B**, the ***vector or "cross " product*** , **A** x **B** is a **vector** whose components are given by the equation

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y\,,\quad A_zB_x - A_xB_z\,,\quad A_xB_y - A_yB_x\,)$$

which is equal to the *determinant*,

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

Main properties of the *vector product*:

- It is ***anti-commutative***, $\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$
- It is ***distributive***, $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
- the resultant is a ***vector*** . Its ***magnitude*** is given by;

$$\boxed{|\,\mathbf{A} \times \mathbf{B}\,| = |A|\;|B|\;\sin\theta}$$

where θ is the **smallest** angle between **A** and **B**.

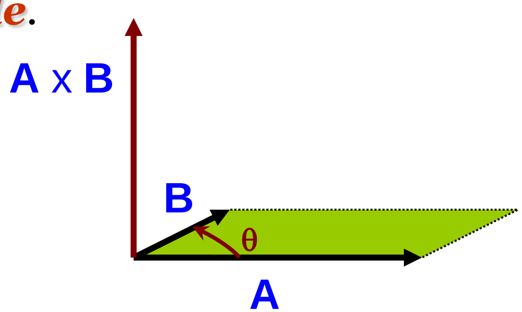
The Vector Product

- The ***direction*** of the resultant vector is $\perp$ to the plane containing **A** and **B**.

Hence,

$$\boxed{\mathbf{A} \times \mathbf{B} = (A\,B\,\sin\theta)\,\mathbf{n}}$$

where **n** is a unit vector normal to the plane containing **A** and **B**. The sense of **n** is given by the ***right –hand rule***.



Therefore,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Note:

The *cross product* $\mathbf{A} \times \mathbf{B}$ has

1- A magnitude of $A\,B\,\sin\theta$ which is equal to ***the area of the parallelogram*** with sides **A** and **B** shown by the shaded area in the Figure.

2- A direction $\perp$ to the plane containing **A** and **B**.

Triple Products

The expression

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

represents the ***scalar triple product*** of **A**, **B** and **C.**
While, the expression

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$$

represents the ***vector triple product*** of **A**, **B** and **C.** It can be remembered easily as the " **back minus cab**" rule.

Derivative of a vector

Previously we were concerned mainly with vector algebra. Now, we will begin to study the calculus of vectors and its use in the description of the motion of particles.

Consider a vector **A**, whose components are functions of a single variable u which is usually the time t, i.e,

$$\mathbf{A}(\mathrm{u}) = \mathbf{i}\, A_x(\mathrm{u}) + \mathbf{j}\, A_y(\mathrm{u}) + \mathbf{k}\, A_z(\mathrm{u})$$

The derivative of **A** with respect to u is defined by

$$\boxed{\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}}$$

Examples:

$$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \qquad \frac{d}{du}(\mathbf{A}.\mathbf{B}) = \frac{d\mathbf{A}}{du}.\mathbf{B} + \mathbf{A}.\frac{d\mathbf{B}}{du}$$

$$\frac{d}{du}(n\mathbf{A}) = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du} \qquad \frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{du}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{du}$$

Notice that it is necessary to maintain the order of the terms in the derivative of the cross product.

Velocity & Acceleration In rectangular coordinates

The position vector

In a given reference system, ***the position of a particle*** can be specified by a single vector. This vector is called ***the position vector*** of the particle.

In rectangular coordinates, the *position vector* is simply

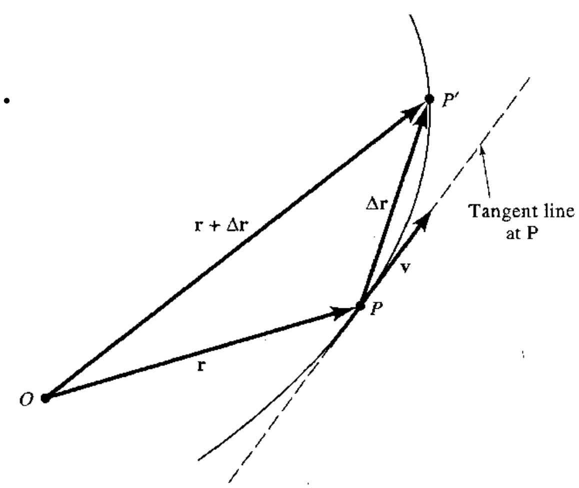
$$\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$$

The velocity vector

For a moving particle, these components are functions of the time. ***The time derivative of* r** *is* called the ***velocity***, **(v)**, which is given by;

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\,\dot{x} + \mathbf{j}\,\dot{y} + \mathbf{k}\,\dot{z}$$

The vector d**r**/dt expresses both the *direction* and the *rate* of motion. As Δt approaches zero, the point P' approaches P, and the direction of the vector Δ**r**/Δt approaches the direction of the tangent to the path at P, which is d**r**/dt.



Note: The velocity vector is always tangent to the path of motion.

The **magnitude** of the velocity is called the *speed* (v). In rectangular components the speed is just

$$v = \frac{ds}{dt} = |\mathbf{v}| = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{1/2}$$

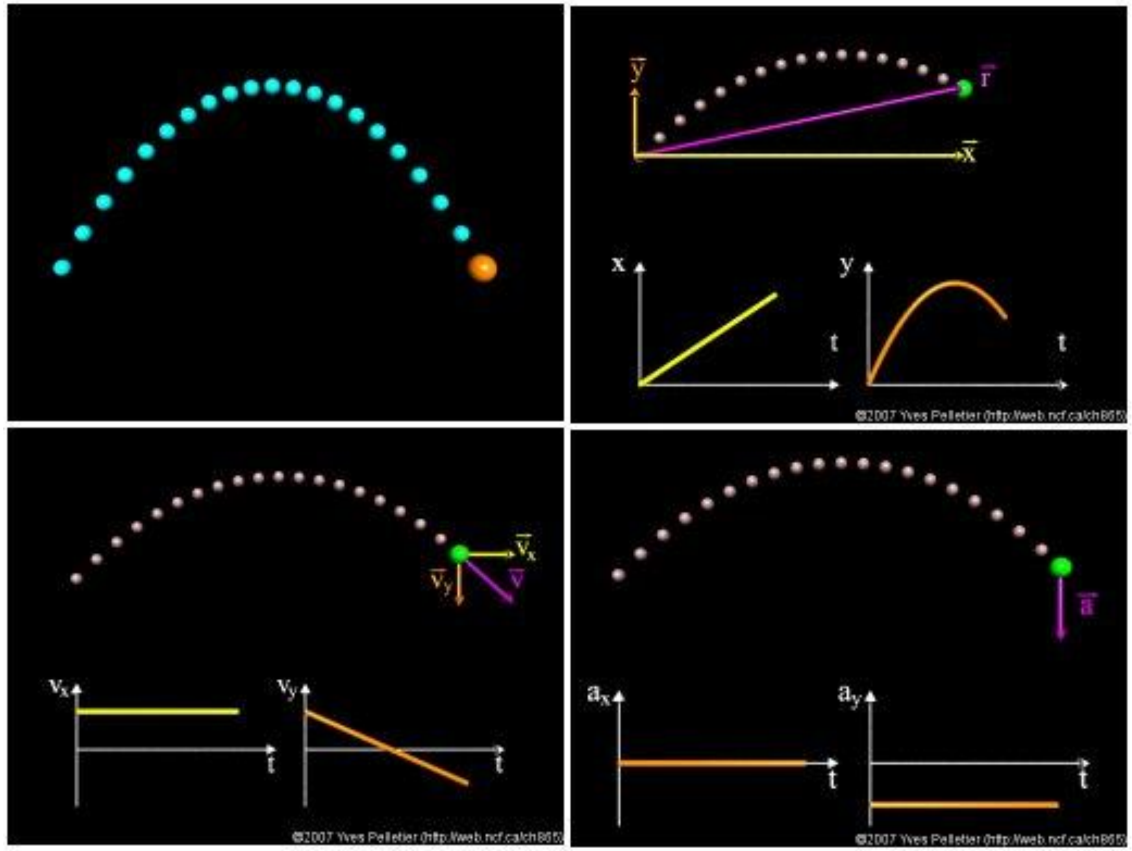
where s is the distance.

The acceleration vector

The time derivative of the velocity is called the ***acceleration*** (**a**). Hence;

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\,\ddot{x} + \mathbf{j}\,\ddot{y} + \mathbf{k}\,\ddot{z}$$

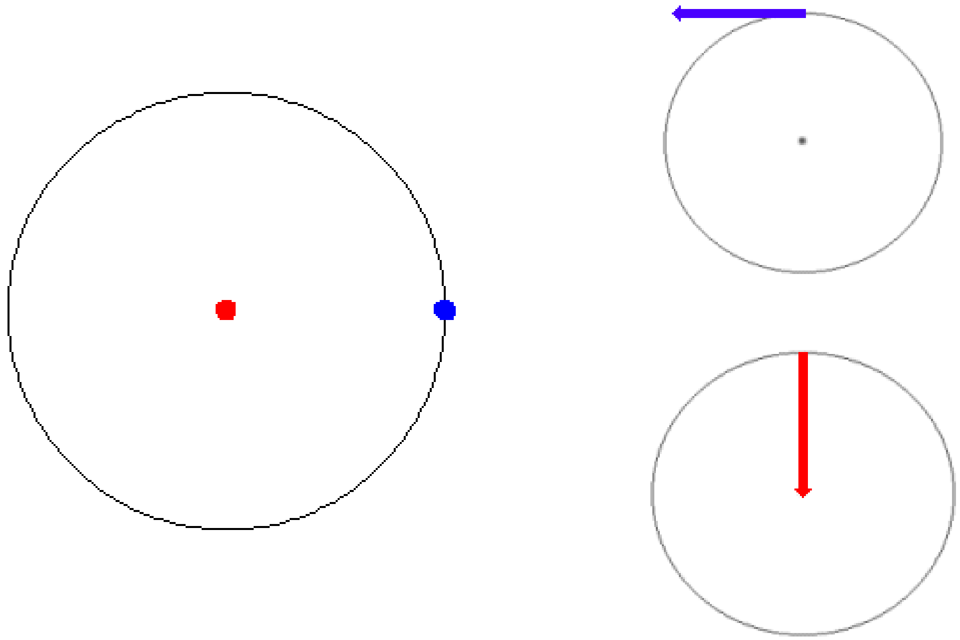
Exp(1-10-1):
Projectile Motion



The position vector	$r = \mathrm{i}\,bt + \mathrm{j}\,(ct - \frac{gt^2}{2})$
The velocity	$\mathbf{v} = \mathbf{i}\,b + \mathbf{j}\,(c - gt)$
The acceleration	$\mathbf{a} = -\mathbf{j}\,g$
The path	**Parabola**

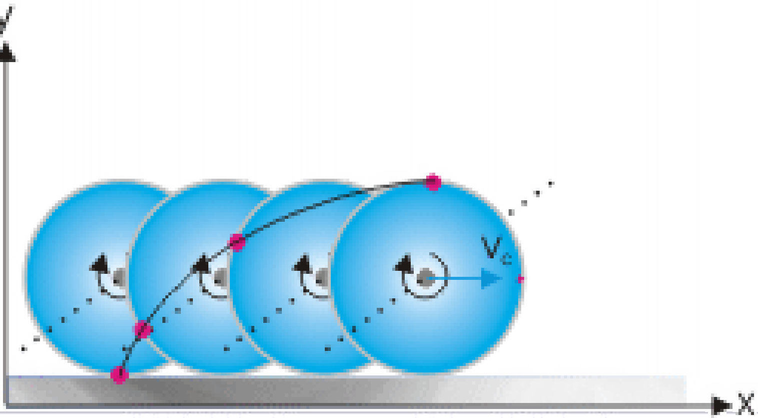
Exp(1-10-2):

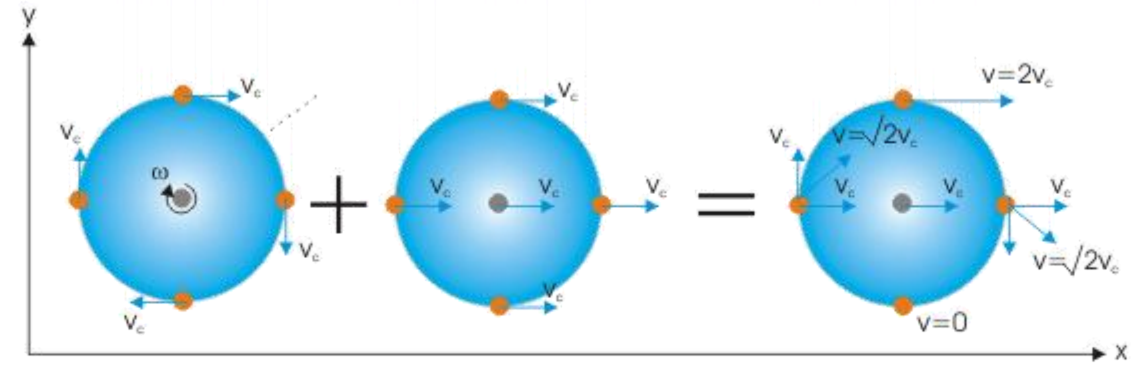
Circular Motion



The position vector	$\mathbf{r} = \mathbf{i}\, b\cos\omega t + \mathbf{j}\, b\sin\omega t$
The velocity	$\mathbf{v} = -\mathbf{i}\, b\omega\, \mathbf{sin}\omega t + \mathbf{j}\, b\omega\, \mathbf{cos}\omega t$
The acceleration	$\mathbf{a} = -\mathbf{i}\, b\omega^2 \cos\omega t - \mathbf{j}\, b\omega^2 \sin\omega t$ $\mathbf{a} = -\omega^2\, \mathbf{r}$
The path	**Circle**

Exp(1-10-3):
Rolling Wheel





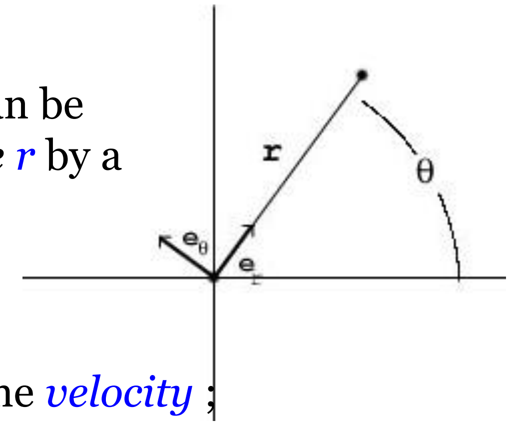
The position vector	$\mathbf{r}_1 = \mathbf{i}\, b\,\omega t + \mathbf{j}\, b$ $\mathbf{r}_2 = \mathbf{i}\, b \sin\omega t + \mathbf{j}\, b\cos\omega t$ $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$
The velocity	$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ $= \mathbf{i}\,(b\omega + b\omega\cos\omega t) - \mathbf{j}\, b\,\omega \sin\omega t$
The acceleration	$\mathbf{a} = -\mathbf{i}\, b\omega^2 \sin\omega t - \mathbf{j}\, b\,\omega^2 \cos\omega t$
The path	**Cycloid**

Velocity & Acceleration in Plane Polar Coordinates

The position vector

The velocity vector

To express the position of a particle moving in a plane, it is often convenient to use *polar coordinates* : r, θ.



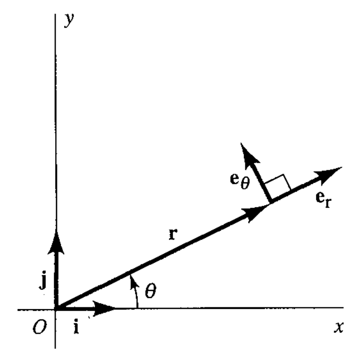
Using Vectors, the position of the particle can be written as the product of the *radial distance* r by a unit *radial vector* $\mathbf{e}_r$:

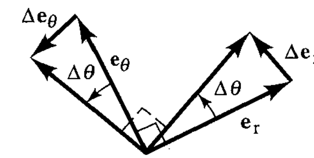
$$\boxed{\mathbf{r} = r\,\mathbf{e}_r}$$

which are both functions of the time.

If we differentiate with respect to t, we get the *velocity* ;

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$





A study of the figure shows that;

$$\Delta\mathbf{e}_r \cong |\Delta\mathbf{e}_r|\ \mathbf{e}_\theta \cong |\mathbf{e}_r|\ \Delta\theta\ \mathbf{e}_\theta \cong \Delta\theta\ \mathbf{e}_\theta$$

$$\Delta\mathbf{e}_\theta \cong -|\Delta\mathbf{e}_\theta|\ \mathbf{e}_r \cong -|\mathbf{e}_\theta|\ \Delta\theta\ \mathbf{e}_r \cong -\Delta\theta\ \mathbf{e}_r$$

Dividing by Δt and taking the limit gives;

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt}$$

Similarly;

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r\frac{d\theta}{dt}$$

The acceleration vector

Hence, $$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta$$

the *radial* component the *transverse* component

To find the ***acceleration vector***, we take the derivative of the velocity with respect to time. This gives

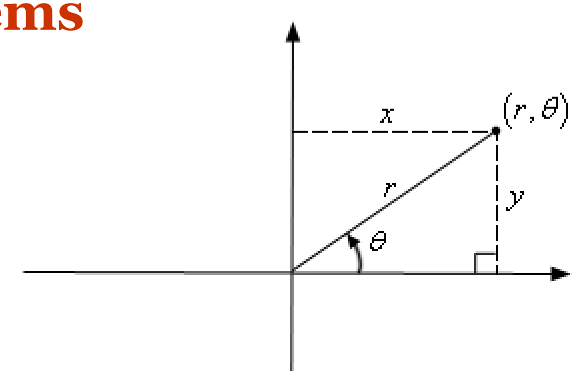
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

the *radial* component the *transverse* component

Note:

Path	r	θ	dr/dt	dθ/dt	a_r	a_θ
Circle	const. (b)	vari.	0	$\dot{\theta}$	$b\dot{\theta}^2$ toward the center	$b\ddot{\theta}$ $\perp$ **r**
Radial line	vari.	const.	$\dot{r}$	0	$\ddot{r}$	0

Converting between Polar & Cartesian coordinate systems



Polar to Cartesian Conversion Formulas

$$x = r\cos\theta \qquad y = r\sin\theta$$

Cartesian to Polar Conversion Formulas

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example

A- Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

B- Convert (-1,-1) into Polar coordinates.

Solution

A-

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$

B-

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4} \Longrightarrow \theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in Polar coordinates this point is $\left(\sqrt{2}, \frac{5\pi}{4}\right)$

Problem

Convert each of the following into an equation in the given coordinate system

A- $2x - 5x^3 = 1 + xy$ into polar coordinates.

B- $r = -8\cos\theta$ into Cartesian coordinates

Solution

A- $2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$

$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

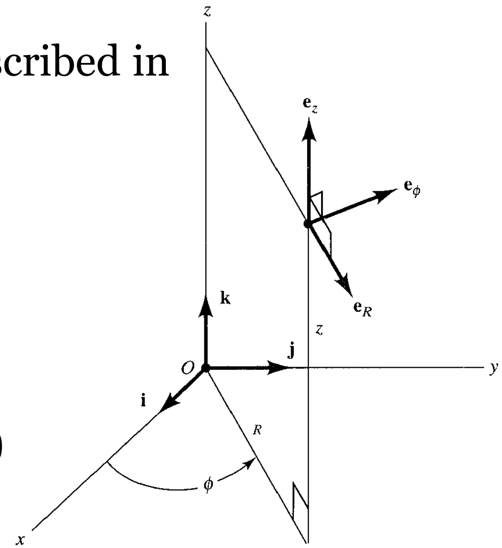
B- $r^2 = -8r\cos\theta$

$$\Longrightarrow \quad x^2 + y^2 = -8x$$

Velocity & Acceleration in Cylindrical Coordinates

The position vector

The *position vector* of a particle can be described in *cylindrical coordinates* (R, ϕ, z) as



$$\boxed{\mathbf{r} = R\mathbf{e}_R + z\mathbf{e}_z}$$

In view of the facts that

$$\frac{d\mathbf{e}_R}{dt} = \dot{\phi}\,\mathbf{e}_\phi \quad , \quad \frac{d\mathbf{e}_\phi}{dt} = -\dot{\phi}\,\mathbf{e}_R \quad \text{and} \quad \frac{d\mathbf{e}_z}{dt} = 0$$

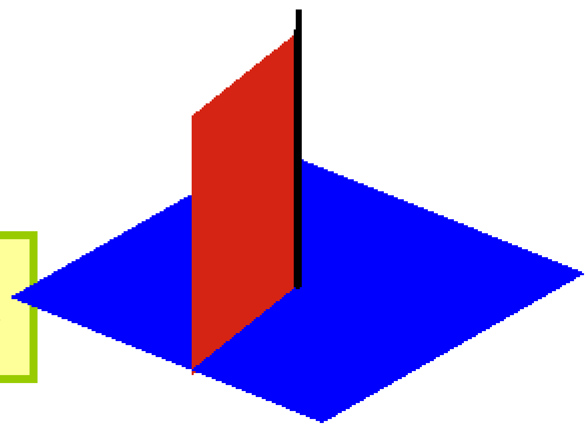
The *velocity* and *acceleration* vectors can be given by the following equations:

The velocity vector

$$\boxed{\mathbf{v} = \dot{R}\mathbf{e}_R + R\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\mathbf{e}_z}$$

The acceleration vector

$$\boxed{\mathbf{a} = \left(\ddot{R} - R\dot{\phi}^2\right)\mathbf{e}_R + \left(R\ddot{\phi} + 2\dot{R}\dot{\phi}\right)\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z}$$



Converting between Cylindrical & Cartesian coordinate systems

Cylindrical to Cartesian Conversion Formulas

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Cartesian to Cylindrical Conversion Formulas

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z$$

Example

A- Convert $(2, \pi/4, 3)$ into Cartesian coordinates.

B- Convert $(1, -1, 2)$ into cylindrical coordinates.

Solution

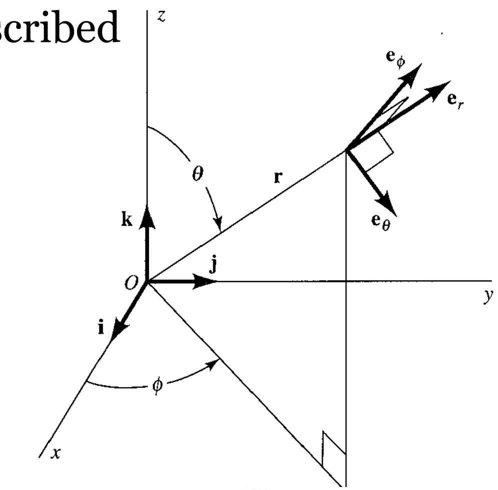
A- In Cartesian coordinates this point is $(2\sqrt{2}, 2\sqrt{2}, 3)$

B- In Polar coordinates this point is $(\sqrt{2}, 7\pi/4, 2)$

Velocity & Acceleration in Spherical Coordinates

The position vector

The *position vector* of a particle can be described in *Spherical coordinates* (r, θ, ϕ) as



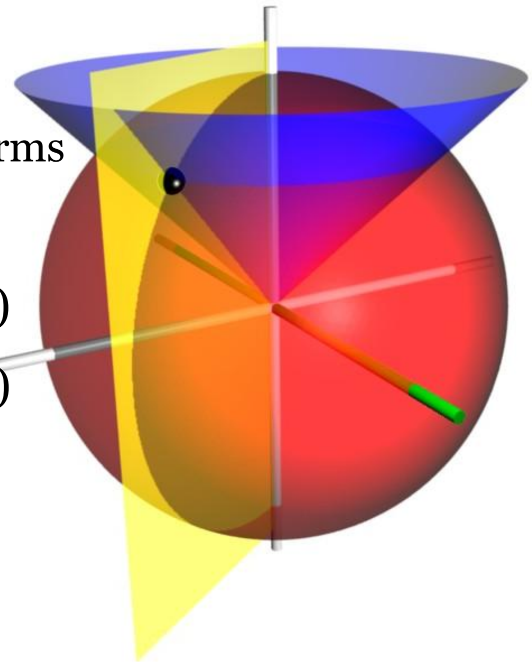
$$\boxed{\mathbf{r} = r\mathbf{e}_r}$$

Note yhat;

$r \geq 0$

$0° \leq \theta \leq 180°$ (π rad)

$0° \leq \varphi < 360°$ (2π rad)



Because any vector can be expressed in terms of its projections onto the x,y,z axes as;

$$\mathbf{e}_r = \mathbf{i}(\sin\theta\cos\phi) + \mathbf{j}(\sin\theta\sin\phi) + \mathbf{k}(\cos\theta)$$

$$\mathbf{e}_\theta = \mathbf{i}(\cos\theta\sin\phi) + \mathbf{j}(\cos\theta\sin\phi) - \mathbf{k}(\sin\theta)$$

$$\mathbf{e}_\phi = -\mathbf{i}(\sin\phi) + \mathbf{j}(\cos\phi)$$

The *velocity* and *acceleration* vectors can be given as the following:

The velocity vector

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\sin\theta\,\mathbf{e}_\phi + r\dot{\theta}\,\mathbf{e}_\theta$$

The acceleration vector

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\mathbf{e}_\theta \\ &+ \left(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta\right)\mathbf{e}_\phi\end{aligned}$$

Converting between Spherical & Cartesian coordinate systems

Spherical to Cartesian Conversion Formulas

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\cos\theta \qquad z = r\cos\theta$$

Cartesian to Spherical Conversion Formulas

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$